Stiffness Matrix $[k]$.

Lets Consider.

$\omega_1, \omega_2, \omega_3 \dots \omega_n \rightarrow$ nodal displacement parameters. or termed as dof

$W_1, W_2, W_3 \dots W_n \rightarrow$ Corresponding nodal loads. acting at doF.

$$\{W\} = \begin{Bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{Bmatrix}, \quad \{\omega\} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{Bmatrix}$$

We know that

$$\{W\} = [k]\{\omega^*\} \rightarrow (1)$$

$W \Rightarrow$ Nodal loads.

$K \Rightarrow$ Stiffness matrix.

$\omega^* \rightarrow$ DOF.

We know that,

Work done, $P =$ Strain energy

$$\Rightarrow P = \tfrac{1}{2} W_1\omega_1 + \tfrac{1}{2} W_2\omega_2 + \tfrac{1}{2} W_3\omega_3 \dots + \tfrac{1}{2} W_n\omega_n$$

$$P = \tfrac{1}{2}\begin{bmatrix} W_1, W_2, W_3 \dots W_n \end{bmatrix}\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_n \end{Bmatrix}$$

$$= \tfrac{1}{2}\left\{ \{W\}^T \{\omega^*\} \right. \rightarrow (2)$$

$[\ ] \rightarrow$ Row matrix

$\{\ \} \rightarrow$ Column matrix.

subst ② in ① we get.

$$P = \frac{1}{2}\left\{[K]\{w^*\}\right\}^T \cdot \{w^*\}$$

$$= \frac{1}{2}[K]^T\{w^*\}^T\{w^*\}$$

$$P = \frac{1}{2}\{w^*\}^T[K]\{w^*\}$$

$$\boxed{P = \frac{1}{2}\{w^*\}^T[K]\{w^*\}} \rightarrow ③$$

K is a symmetric matrix. so. $[K]^T = [K]$

This is strain energy equation for a structure.

We need to find the expression for stiffness matrix $[K]$.

Let us consider one dimensional element.

$u_1, u_2, u_3 \cdots u_n$ are the degrees of freedom of that element.

We know that.

$e = \frac{du}{dx}$

$e = B \times u$

$$\text{Strain}\ \{e\} = [B]\{u^*\}$$

$$\{e\}^T = [B]^T\{u^*\}. \rightarrow ④$$

$\{e\}$ is a strain matrix

$[B] \rightarrow$ strain displacement relation matrix

$\{u\} \rightarrow$ dof.

also we know that

$$\text{Stress}\ \{\sigma\} = [E]\{e\}$$

$$\{\sigma\} = [D]\{e\} \rightarrow ⑤$$

$[E] = [D]$ = Young modulus.

Strain energy expression is given by

$$\boxed{U = \int_V \frac{1}{2} \{e\}^T \{\sigma\} \cdot dV} \rightarrow 6$$

$$\boxed{U = \frac{1}{2} V \cdot \sigma e}$$

$\sigma \rightarrow$ stress.
$e \rightarrow$ strain.
$V \rightarrow$ volume.

Subs e^T(4) & σ(5) in above eqn. 6

$$U = \int_V \cdot \frac{1}{2} [B]^T \{u^*\}^T \cdot [D] \{e\} \cdot dV$$

$$= \frac{1}{2} \{u^*\}^T \int_V [B]^T [D] \{e\} \cdot dV$$

$$U = \frac{1}{2} \{u^*\}^T \int_V [B]^T [D] [B] \{u^*\} \cdot dV$$

$$= \frac{1}{2} \{u^*\}^T \left[\int_V [B]^T [D] [B] \cdot dV \right] \{u^*\} \rightarrow 7$$

Comparing 3 with 7

$$3 \Rightarrow P = \frac{1}{2} \{u^*\}^T [K] \{u^*\}$$

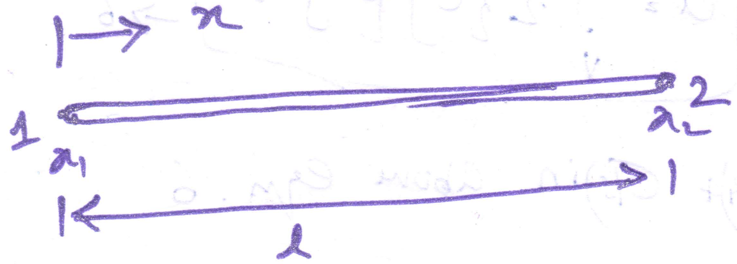
we get.

$$[K] = \int_V [B]^T [D] [B] \cdot dV$$

~~in ADp~~

$$\cancel{\text{Strain } e = \frac{du}{dx}}$$

Derivation of Stiffness matrix for one Dimensional Bar element.



a bar element with ~~tw~~ two nodes.

Stiffness matrix $[K] = \int_V [B]^T [D] [B] \cdot dV$

for 1D elem we know

$u = N_1 u_1 + N_2 u_2$ where $N_1 = \frac{l-x}{l}$, $N_2 = \frac{x}{l}$.

$$[B] = \left[\frac{dN_1}{dx} \;,\; \frac{dN_2}{dx} \right]$$

$$[B] = \left[\frac{-1}{l} \quad \frac{1}{l} \right]$$

$$[B]^T = \begin{Bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{Bmatrix}$$

Sub the val of $[B^T]\,[B]$ in Stiffness matrix equation

$$[K] = \int_0^V \begin{Bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{Bmatrix} \times E \times \left[\frac{-1}{l} \cdot \frac{1}{l} \right] \cdot dV$$

$$= \int_0^V \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} E \cdot dV$$

$$= \frac{1}{l^2}\int_0^l \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot EA \cdot dx$$

$$= \frac{1}{l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} EA \, [x]_0^l$$

$$= \frac{1}{l^{\cancel{2}}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} AE\cancel{l}$$

$$[k] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Properties of a Stiffness matrix.

(i) is asymmetric matrix

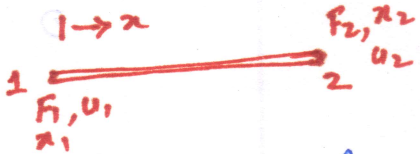
(2) The Sum of elements in any column is equal to zero.

(3) Its an unstable element, so the determinant is zero

(4) The dimension of the global stiffness matrix [k] is N×N, where N is the no. of nodes.

(5) The diagonal coeffn are always positive and relatively large when compared to the off diagonal values in the same row.

Derivation of Finite Element Eqn for One dimensional Bar Element



General form eqn is

$$\{F\} = [K]\{u\}$$

$\{F\}$ → elem force
$[K]$ stiffn mat.
$\{u\}$ nodal displ

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}, \quad \{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{AE}{L}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Thus is a FE eqn for 1D. 2 noded bar element.